The classification of root systems

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1 Introduction

An irreducible root system is a finite set of vectors in Euclidean space satisfying certain properties. The goal of this essay is to classify all irreducible root systems. I mostly follow the books given in references, however some information is from other sources, such as *Wikipedia*, *PlanetMath*, and *MathWorld*.

2 Root systems

Definition. Let $\mathbb{E} \cong \mathbb{R}^n$ be a vector space with an inner product $\langle \cdot, \cdot \rangle$. A subset $R \subset \mathbb{E} \setminus \{0\}$ is called *root system*, if R has the following properties:

(R1) R is finite and spans $\mathbb{E}$,

(R2) if $\alpha \in R$, then $-\alpha \in R$ and $\pm\alpha$ are the only multiples of α in R,

(R3) R is invariant under the reflection in the hyperplane orthogonal to any $\alpha \in R$ (see Fig. 1), i.e., for all $\alpha, \beta \in R$:

$$s_\alpha(\beta) = \beta - 2 \operatorname{proj}_\alpha \beta \in R, \tag{1}$$

where $\operatorname{proj}_\alpha \beta$ is the projection of β on α:

$$\operatorname{proj}_\alpha \beta = \alpha \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}, \tag{2}$$

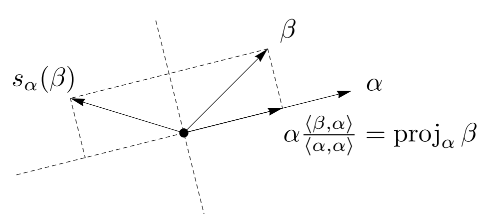

Figure 1: The reflection $s_\alpha(\beta)$ of β in the hyperplane orthogonal to α.

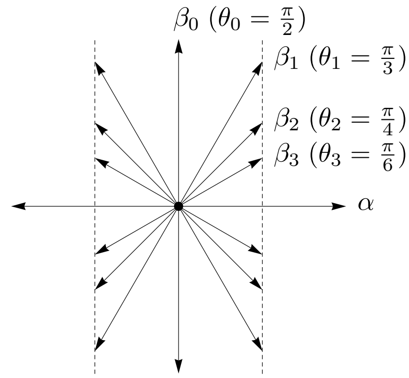

Figure 2: The possible directions for β, when α is fixed.

(R4) R is *crystallographic*, i.e., for all $\alpha, \beta \in R$:

$$n_{\beta\alpha} = 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}. \tag{3}$$

The elements of R are called *roots* and the dimension of $\mathbb{E}$ is called the *rank* of the root system.

Definition. The root system R is called *decomposable* if there is a proper decomposition $R = R_1 \cup R_2$ such that $\forall \alpha_1 \in R_1, \forall \alpha_2 \in R_2 : \langle \alpha_1, \alpha_2 \rangle = 0$. Otherwise it is called *indecomposable* or *irreducible*.

The condition $-\alpha \in R$ in property (R2) is not needed, because it follows from (R4), since $s_\alpha(\alpha) = -\alpha$. We can interpret the property (R4) geometrically as follows – the projection of β on α is an integer or half-integer multiple of α, since

$$\operatorname{proj}_\alpha \beta = \frac{1}{2} n_{\beta\alpha} \alpha.$$

In fact, this is the most restrictive property, because

$$n_{\beta\alpha} = 2\frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} = 2\frac{\|\beta\| \, \|\alpha\| \cos\theta}{\|\alpha\|^2} = 2\frac{\|\beta\|}{\|\alpha\|} \cos\theta \in \mathbb{Z},$$

where θ is the angle between α and β. Since both $n_{\beta\alpha}$ and $n_{\alpha\beta}$ are integers:

$$n_{\beta\alpha} \cdot n_{\alpha\beta} = 4\cos^2\theta \in \mathbb{Z}.$$

More precisely, $4\cos^2\theta \in \{0, 1, 2, 3, 4\}$. If $4\cos^2\theta = 4$, then $\theta \in \{0, \pi\}$, which is just the property (R2). The other cases are summarized in Table 1 and the corresponding vectors are shown in Fig. 2.

Let us consider root systems of small rank and see what are the possible configurations that we can get (the examples are taken from [1] and [2]).

2.1 Root systems of rank 1

If we choose any non-zero vector $\alpha \in \mathbb{R}$, then $R = \{\alpha, -\alpha\}$ is a root system. Since any other non-zero vector is a multiple of α, property (R2) forbids us to add more vectors to our root system. Therefore in rank 1 there is only one possible root system – it is called A_1 (see Fig. 3).

$4\cos^2\theta$	$n_{\beta\alpha}$	$n_{\alpha\beta}$	$\|\alpha\| \,/\, \|\beta\|$	$\cos\theta$	θ
3	$+1$	$+3$	$\sqrt{3}$	$+\sqrt{3}/2$	$\pi/6$
	-1	-3	$\sqrt{3}$	$-\sqrt{3}/2$	$5\pi/6$
2	$+1$	$+2$	$\sqrt{2}$	$+\sqrt{2}/2$	$\pi/4$
	-1	-2	$\sqrt{2}$	$-\sqrt{2}/2$	$3\pi/4$
1	$+1$	$+1$	1	$+1/2$	$\pi/3$
	-1	-1	1	$-1/2$	$2\pi/3$
0	0	0	any	0	$\pi/2$

Table 1: Possible values of $4\cos^2\theta$ and the corresponding angles θ. We assume that α is longer than β.

Figure 3: The root system A_1.

2.2 Root systems of rank 2

In rank 2 there is more freedom, because we can use any angle θ given in Table 1. The simplest root system corresponds to $\theta = \pi/2$. It is called $A_1 \times A_1$, because it is a direct sum of two rank 1 root systems A_1 (see Fig. 4). Therefore it is decomposable and the ratio of lengths of vertical and horizontal roots can be arbitrary.

When $\theta = \pi/3$, the root system consists of 6 vectors that correspond to the vertices of a regular hexagon. This root system is called A_2 and it is shown in Fig. 5 (the purpose of the dashed lines is to indicate the lengths of projections as in [3, pp. 120]).

If $\theta = \pi/4$, the root system consists of 8 vectors. They correspond to the vertices and to the midpoints of the edges of a regular square (see Fig. 6). The ratio of lengths of these roots is $\sqrt{2}$. This root system is called B_2.

Finally, if $\theta = \pi/6$, the root system consists of 12 vectors. They correspond to the vertices of two regular hexagons that have different sizes and are rotated away from each other by an angle $\pi/6$ (see Fig. 7). The ratio of lengths of these vectors is $\sqrt{3}$. This is an "exceptional" root system and is called G_2.

It is not hard to see, that there are no other root systems of rank 2, because in two dimensions the angle θ determines the root system completely, i.e., once

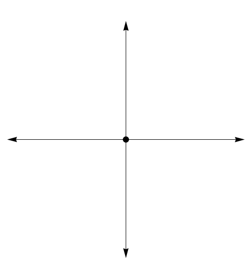

Figure 4: The root system $A_1 \times A_1$.

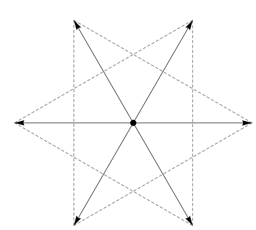

Figure 5: The root system A_2.

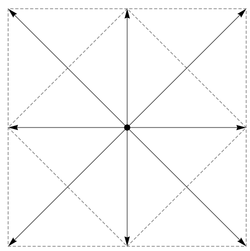

Figure 6: The root system B_2.

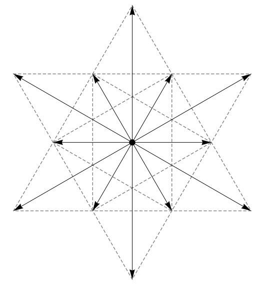

Figure 7: The root system G_2.

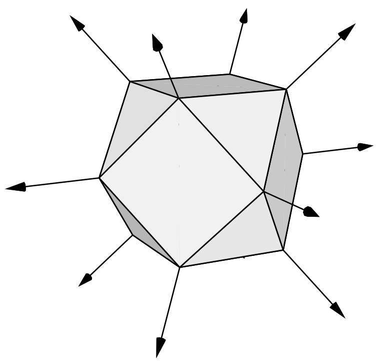

Figure 8: The root system A_3.

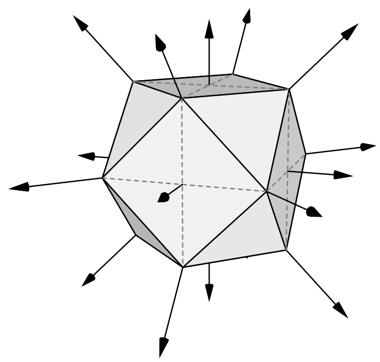

Figure 9: The root system B_3.

the angle is chosen, the ratio of lengths of two consecutive roots is determined (except for the case $\theta = \pi/2$), hence the root system itself.

2.3 Root systems of rank 3

In rank 3 there are more decomposable root systems than in rank 2, because we can use any root system of a lower rank to build one with a higher rank. The decomposable root systems are: $A_1 \times A_2$, $A_1 \times B_2$, $A_1 \times G_2$, and $A_1 \times A_1 \times A_1$. But there are also three irreducible root systems.

The smallest irreducible root system of rank 3 consists of 12 points and is called A_3 (see Fig. 8). These roots correspond to the vertices of a regular *cuboctahedron* (the intersection of a *cube* and an *octahedron*). One can think of cuboctahedron as a cube with corners cut off. Then the roots correspond to the midpoints of the edges of the cube. It means, they have the same length.

We can extend this root system by adding six vectors that are $\sqrt{2}$ times shorter and correspond to the midpoints of the quadrangular faces of the cuboctahedron or simply to the faces of the cube (see Fig. 9). The obtained root system has 18 vectors and is called B_3.

It turns out that we can extend A_3 in another way. We use the same six vectors, but this time we take them to be $\sqrt{2}$ times longer than the ones already in A_3 (see Fig. 10). This gives us a different kind of root system that also consists of 18 vectors and is called C_3. It looks different, because the convex hull of the roots is an octahedron. But one can still see the cuboctahedron

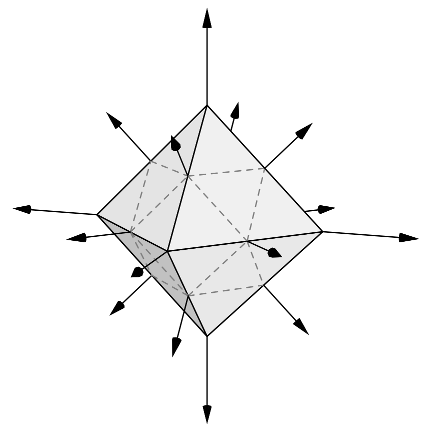

Figure 10: The root system C_3.

behind it (consider the dashed lines in Fig. 10, that join the midpoints of the edges of the octahedron).

The root systems A_3, B_3, and C_3 are the only irreducible root systems of rank 3 (see [1, pp. 323] and [2, pp. 163, 262]). Since the root systems of rank 4 will not be easy to visualize, let us proceed to the classification of root systems of any rank.

3 Classification of root systems

The proof of the classification theorem can be found in several textbooks, e.g., [1, pp. 325], [4, pp. 186], [5, pp. 130], [6, pp. 57], and [7, pp. 201]. I will follow the proofs given in [5] and [6].

3.1 Simple roots

For each root system one can choose a special subset (though it is not unique) of roots called *simple roots* or *fundamental system.* It plays a very important role in the classification of irreducible root systems.

Consider a root system R. For each root there is a unique hyperplane that contains the origin and is orthogonal to this root. Since a root system is finite, the union of all such hyperplanes can not be the whole space. Thus one can find a vector d, such that $\forall\alpha \in R : \langle\alpha, d\rangle \neq 0$. Then we can break the root system into two disjoint parts $R = R^+(d) \cup R^-(d)$, where $R^+(d) = \{\alpha \in R | \langle\alpha, d\rangle > 0\}$ and $R^-(d) = -R^+(d)$.

Definition. A root α is called *positive* if $\alpha \in R^+(d)$ and *negative* if $\alpha \in R^-(d)$.

Definition. A positive root $\alpha \in R^+(d)$ is called *simple* if it is not a sum of two other positive roots.

Definition. The set of all simple roots of a root system R is called *basis* or *fundamental system* of R.

One might think that different choices of d can lead to differently looking bases, but it turns out that this is not the case. For each root $\alpha \in R$ there is a hyperplane orthogonal to α and the union of all these hyperplanes cut the space $\mathbb{E}$ into open, connected regions called *Weyl chambers*. It turns out that there is a one-to-one correspondence between bases and *Weyl chambers*:

- given a basis $\Delta \subset R$ the corresponding Weyl chamber C consists of all vectors in $\mathbb{E}$ having positive inner product with all simple roots from Δ,
- given a Weyl chamber C those $\alpha \in R$ that have positive inner product with all vectors from C are positive roots R^+ and they determine the set of simple roots Δ.

Definition. The group generated by reflections s_α is called *Weyl group.*

Lemma 1. *Any two bases of a given root system $R \subset \mathbb{E}$ are equivalent under the action of the Weyl group.*

Moreover, it turns out that the basis of a root system contains all information about it, i.e., knowing simple roots is enough to recover the whole root system.

Lemma 2. *The root system R can be uniquely reconstructed from its basis.*

This reconstruction is done by repeatedly applying reflections (1) in the hyperplanes orthogonal to the simple roots.

3.2 Properties of simple roots

Lemma 3. *If $\alpha, \beta \in R$ are not proportional and $\langle \alpha, \beta \rangle > 0$, then $\alpha - \beta \in R$.*

Proof. Since $n_{\alpha\beta} = 2 \langle \alpha, \beta \rangle / \langle \beta, \beta \rangle > 0$, from Table 1 we see that either $n_{\alpha\beta}$ or $n_{\beta\alpha}$ is 1. If $n_{\alpha\beta} = 1$, then $s_\beta(\alpha) = \alpha - 2\beta \langle \alpha, \beta \rangle / \langle \beta, \beta \rangle = \alpha - \beta \cdot n_{\alpha\beta} = \alpha - \beta \in R$. If $n_{\beta\alpha} = 1$, then $s_\alpha(-\beta) = -\beta - \alpha \cdot n_{(-\beta)\alpha} = -\beta + \alpha \cdot n_{\beta\alpha} = \alpha - \beta \in R$. $\square$

Lemma 4. *If α and β are distinct simple roots, then $\langle \alpha, \beta \rangle \leq 0$.*

Proof. Assume the opposite, i.e., $\langle \alpha, \beta \rangle > 0$. Then from the previous lemma we have $\alpha - \beta = \gamma \in R$. If $\gamma \in R^+(d)$, then $\alpha = \beta + \gamma \in R^+(d)$, which contradicts that α is simple. Otherwise $-\gamma \in R^+(d)$ and $\beta = \alpha + (-\gamma) \in R^+(d)$, which contradicts that β is simple. $\square$

We will need two more properties of simple roots:

Lemma 5. *The simple roots are linearly independent and span the whole space.*

Lemma 6. *The root system is decomposable if and only if its base is decomposable.*

3.3 The classification theorem

According to Lemma 4 two simple roots α and β are either orthogonal or the angle θ between them is obtuse. From Table 1 we see that θ is either $\pi/2$, $2\pi/3$, $3\pi/4$ or $5\pi/6$. We can encode this information into a graph:

Definition. The *Coxeter graph* of a root system R is a graph that has one vertex for each simple root of R and every pair α, β of distinct vertices is connected by $n_{\alpha\beta} \cdot n_{\beta\alpha} = 4\cos^2\theta \in \{0, 1, 2, 3\}$ edges (hence it is a multigraph).

However, there is some information missing, namely the relative lengths of the roots. If the angle between two roots is either $5\pi/6$ or $3\pi/4$, then we have to specify which root is longer. If the angle is $2\pi/3$, then both roots have the same length, so we do not need to add anything. Finally, since we are interested only in irreducible root systems, the lengths of roots with the angle $\pi/2$ must be correlated (only if the root system is decomposable into two mutually orthogonal sets, we can rescale the roots in one of them by an arbitrary factor, without affecting the roots in other sets), therefore in this case we are also done. Hence we can encode all the information as follows:

Definition. The *Dynkin diagram* of a root system is its Coxeter graph with arrows attached to the double and triple edges that point to the shorter root.

In fact, to classify the irreducible root systems, it is enough to consider only the Coxeter graphs. Once all possible Coxeter graphs are known, we can add arrows to obtain the corresponding Dynkin diagrams. Therefore we will forget about the lengths of the roots for now.

Definition. A linearly independent set of n unit vectors $\{v_1, v_2, \ldots, v_n\}$ that spans $\mathbb{E}$ is called an *admissible configuration* if for all $i \neq j : \langle v_i, v_j \rangle \leq 0$ and $4\langle v_i, v_j \rangle^2 = 4\cos^2\theta \in \{0, 1, 2, 3\}$.

Note the set of normalized simple roots of any root system is an admissible configuration, since according to Lemma 5 they are linearly independent and span the whole space.

Definition. Coxeter graph of an admissible configuration is *admissible diagram.*

According to Lemma 6 the set of simple roots of an irreducible root system can not be decomposed into mutually orthogonal subsets. It means that the corresponding Coxeter graph will be connected. Thus, to classify all irreducible root systems, we will consider only connected admissible diagrams. Now we are ready to prove the classification theorem.

Theorem. *The Dynkin diagram of an irreducible root system is one of the diagrams shown in Fig. 11.*

Proof. Let us classify the connected admissible diagrams first and then proceed to the Dynkin diagrams. The classification consists of several steps:

1. *Any subdiagram of an admissible diagram is also admissible.* If the set $\{v_1, v_2, \ldots, v_n\}$ is an admissible configuration, then clearly any subset of it is also an admissible configuration (in the space it spans). The same holds for admissible diagrams.

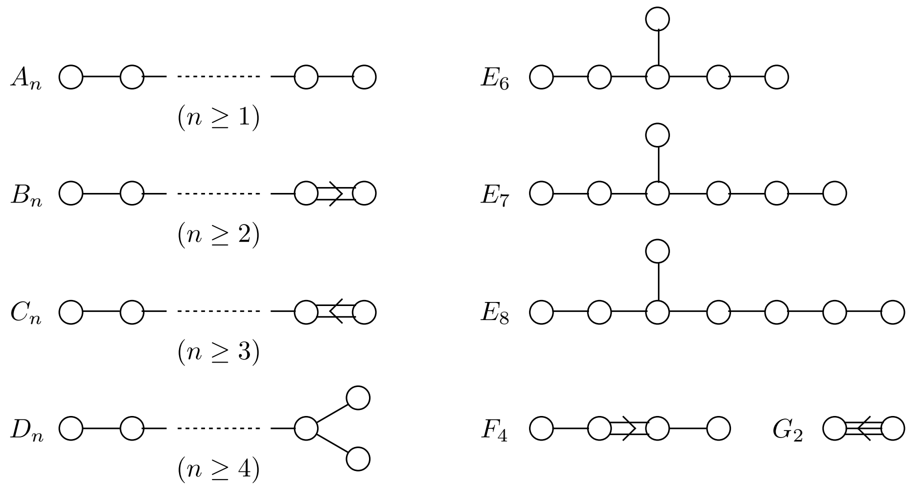

Figure 11: Four infinite families and five exceptional root systems.

2. *A connected admissible diagram is a tree.* Define $v = \sum_{i=1}^{n} v_i$. It is clear that $v \neq 0$, since the vectors $v_1, v_2, \ldots, v_n$ are linearly independent. Then

$$0 < \langle v, v \rangle = \sum_{i=1}^{n} \langle v_i, v_i \rangle + \sum_{i<j} 2 \langle v_i, v_j \rangle = n + \sum_{i<j} 2 \langle v_i, v_j \rangle .$$

If the vertices v_i and v_j are connected, then $2 \langle v_i, v_j \rangle \in \left\{-1, -\sqrt{2}, -\sqrt{3}\right\}$. In particular, $2 \langle v_i, v_j \rangle \leq -1$. It means, the number of terms in the sum can not exceed $n-1$, thus the number of distinct pairs of connected vertices is also at most $n-1$. Since the diagram is connected, there must be at least $n-1$ such pairs. Therefore the number of distinct connected pairs of vertices is exactly $n-1$ and the diagram is a tree.

3. *No more than three edges (counting multiplicities) can originate from the same vertex.* Let c be any vertex and $v_1, v_2, \ldots, v_k$ be all vertices that are connected to c. Since the graph has no cycles, there are no edges between any v_i and v_j. Thus $\langle v_i, v_j \rangle = 0$ when $i \neq j$ and $\{v_1, v_2, \ldots, v_k\}$ is an orthonormal set. Since the simple roots are linearly independent, c can not be expressed as a linear combination of v_i's. Hence c has a non-zero projection to the orthogonal complement of $\operatorname{span}\{v_1, v_2, \ldots, v_k\}$. Let us normalize this projection and denote it by v_0. Then $\{v_0, v_1, v_2, \ldots, v_k\}$ is an orthonormal set and we can express c as follows:

$$c = \sum_{i=0}^{k} \langle c, v_i \rangle v_i .$$

Since c is a unit vector, $\langle c, c \rangle = \sum_{i=0}^{k} \langle c, v_i \rangle^2 = 1$. But $\langle c, v_0 \rangle \neq 0$, thus

$$\sum_{i=1}^{k} 4 \langle c, v_i \rangle^2 < 4. \tag{4}$$

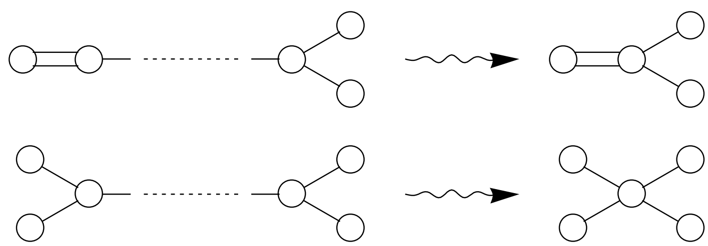

Figure 12: Collapsing simple chains to obtain forbidden subdiagrams.

The quantity $4\left\langle c, v_i\right\rangle^2$ is the number of edges between c and v_i, thus from (4) it follows that the number of edges originating at c is less than 4.

4. *The only connected admissible diagram containing a triple edge is G_2 that is shown in Fig. 11.* This follows from the previous step. From now on we will consider only diagrams with single and double edges.

5. *Any simple chain $v_1, v_2, \ldots, v_k$ in a connected admissible diagram can be replaced by a single vector $v = \sum_{i=1}^{k} v_i$.*

   Definition. A *simple chain* is a non-repeating sequence of vertices such that every two consecutive vertices are connected with a single edge.

   We must show that v is a unit vector and the obtained diagram is admissible and connected. We have:

$$\langle v, v\rangle = k + \sum_{i<j} 2\left\langle v_i, v_j\right\rangle .$$

   There are no cycles, thus $\left\langle v_i, v_j\right\rangle = 0$ for all pairs $i < j$, except $j = i+1$. For two consecutive vertices in the chain we have $2\left\langle v_i, v_{i+1}\right\rangle = -1$, thus

$$\sum_{i<j} 2\left\langle v_i, v_j\right\rangle = \sum_{i=1}^{k-1} 2\left\langle v_i, v_{i+1}\right\rangle = -(k-1)$$

   and $\langle v, v\rangle = k - (k-1) = 1$, hence v is a unit vector.

   Since there are no cycles, an arbitrary vertex u that is not in the chain, can be connected to at most one vertex (let it be v_j) in the chain. Then

$$\langle u, v\rangle = \sum_{i=1}^{k} \left\langle u, v_i\right\rangle = \left\langle u, v_j\right\rangle .$$

   It means, the whole chain is replaced by a single vertex v and any vertex u not in the chain remains connected to v in the same way it was connected to v_j. Therefore the obtained diagram is also admissible and connected.

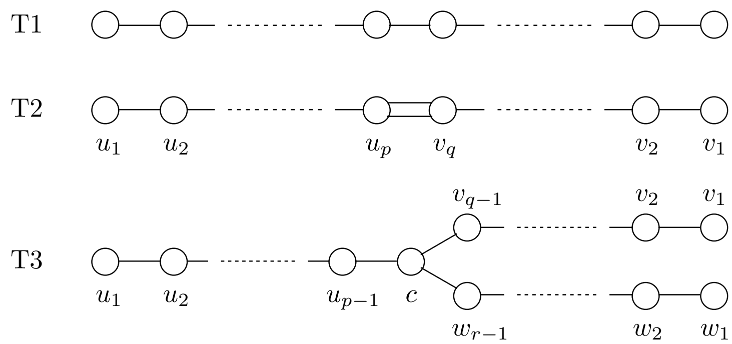

Figure 13: Three possible types of connected admissible diagrams.

6. *A connected admissible diagram has none of subdiagrams shown in Fig. 12.* In each case the subdiagram contains a simple chain. According to Step 5 it can be collapsed to a single vertex. But according to Step 3 the obtained subdiagram is not valid, since it has a vertex of degree four. This is a contradiction with Step 1.

7. It means that a connected admissible diagram can contain at most one double edge and at most one branching, but not both of them simultaneously. If we neglect the diagram G_2 with a triple edge, we can make the following conclusion. *There are only three possible types of connected admissible diagrams (see Fig. 13):*

   T1: *a simple chain,*

   T2: *a diagram with a double edge,*

   T3: *a diagram with branching.*

8. *The admissible diagram of type* T1 *corresponds to the Dynkin diagram* A_n *in Fig. 11, where* $n \geq 1$.

9. *The only admissible diagrams of type* T2 *are* $B_n = C_n$*, and* F_4. Define $u = \sum_{i=1}^{p} i \cdot u_i$. Since $2\langle u_i, u_{i+1}\rangle = -1$ for $1 \leq i \leq p-1$, we get

$$\begin{aligned} \langle u, u\rangle &= \sum_{i=1}^{p} i^2 \langle u_i, u_i\rangle + \sum_{i<j} ij \cdot 2\langle u_i, u_j\rangle \\ &= \sum_{i=1}^{p} i^2 + \sum_{i=1}^{p-1} i(i+1) \cdot 2\langle u_i, u_{i+1}\rangle \\ &= \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = p^2 - \sum_{i=1}^{p-1} i \\ &= p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2}. \end{aligned} \tag{5}$$

In a similar way we define $v = \sum_{j=1}^{q} j \cdot v_j$ and get $\langle v, v\rangle = q(q+1)/2$. Finally, $\langle u, v\rangle = pq\langle u_p, v_q\rangle$, because the double edge is the only edge

p	q	r	Dynkin diagram
any	2	2	D_n
3	3	2	E_6
4	3	2	E_7
5	3	2	E_8

Table 2: Possible integer solutions of inequality (8) and the corresponding Dynkin diagrams of type T3.

between u_i's and v_j's. Thus $4\langle u_p, v_q\rangle^2 = 2$ and we get $\langle u, v\rangle^2 = p^2q^2/2$. Since u is not a multiple of v, Cauchy-Schwarz inequality holds strictly: $\langle u, v\rangle^2 < \langle u, u\rangle \langle v, v\rangle$. Therefore

$$\frac{p^2q^2}{2} < \frac{p(p+1)}{2} \cdot \frac{q(q+1)}{2}.$$

Since p and q are positive integers, we get $2pq < (p+1)(q+1)$ or equivalently $(p-1)(q-1) < 2$. The only solutions are $p = q = 2$ or $p = 1$ and q is arbitrary (or vice versa).

The first solution corresponds to the Dynkin diagram F_4 in Fig. 11. The second solution corresponds either to the Dynkin diagram B_n or to C_n (we have to choose the direction of the arrow on the double edge). If $n = 1$, both diagrams coincide with A_1, but if $n = 2$, we have $B_2 = C_2$. Therefore we can use a convention that B_n has $n \geq 2$, but C_n has $n \geq 3$.

10. *The only admissible diagrams of type* T3 *are D_n, E_6, E_7, and E_8.* As before, define $u = \sum_{i=1}^{p-1} i \cdot u_i$, $v = \sum_{j=1}^{q-1} j \cdot v_j$, and $w = \sum_{k=1}^{r-1} k \cdot w_k$. Since there are no direct edges between u_i's, v_j's, and w_k's, they are in mutually orthogonal subspaces. The same holds for u, v, and w. Using a similar argument as in Step 3 we conclude that c is not a linear combination of u, v, and w, therefore

$$1 = \langle c, c\rangle > \langle c, u'\rangle^2 + \langle c, v'\rangle^2 + \langle c, w'\rangle^2, \tag{6}$$

where $u' = u/\sqrt{\langle u, u\rangle}$, $v' = v/\sqrt{\langle v, v\rangle}$, and $w' = w/\sqrt{\langle w, w\rangle}$ are the unit vectors in directions u, v, and w. Thus

$$\langle c, u'\rangle^2 = \frac{\langle c, u\rangle^2}{\langle u, u\rangle}. \tag{7}$$

None of u_i is connected to c, except u_{p-1}, thus $\langle c, u_i\rangle^2 = 0$ unless $i = p-1$. For u_{p-1} we have $4\langle c, u_{p-1}\rangle^2 = 1$, because c and u_{p-1} are connected with a single edge. Therefore the numerator of (7) is

$$\langle c, u\rangle^2 = \sum_{i=1}^{p-1} i^2 \langle c, u_i\rangle^2 = (p-1)^2 \langle c, u_{p-1}\rangle^2 = \frac{(p-1)^2}{4}.$$

According to (5), the denominator of (7) is $p(p-1)/2$, thus (7) becomes

$$\langle c, u'\rangle^2 = \frac{(p-1)^2}{4} \cdot \frac{2}{p(p-1)} = \frac{p-1}{2p} = \frac{1}{2}\left(1 - \frac{1}{p}\right),$$

where $p-1$ was canceled out, since $p \geq 2$. If we do the same for v' and w', equation (6) becomes $2 > (1-1/p)+(1-1/q)+(1-1/r)$ or simply

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1, \quad p,q,r \geq 2. \tag{8}$$

We can assume that $p \geq q \geq r \geq 2$. There is no solution with $r \geq 3$, since then the left hand side of (8) can not exceed 1. Therefore we have to take $r=2$. If we take $q=2$ as well, then any p suits, but for $q=3$ we have $1/q+1/r=5/6$ and we can take only $p<6$. There are no solutions with $q \geq 4$, because then the left hand sinde of (8) is at most 1. The corresponding type T3 Dynkin diagrams are summarized in Table 2.

This completes the classification theorem. □

We have shown that for any root system the corresponding Dynkin diagram is one of the diagrams in Fig. 11, but it does not mean that for each diagram there indeed is a corresponding root system. However, for each diagram shown in Fig. 11 there indeed is a root system whose diagram it is. The construction of all root systems can be found in several textbooks, e.g., [1, pp. 330], [5, pp. 135], [6, pp. 63], and [8, pp. 293]. Therefore

Theorem. *For each Dynkin diagram shown in Fig. 11 there is an irreducible root system having the given diagram.*

References

[1] Fulton W., Harris J., *Representation Theory: A First Course*, Springer, 1991.

[2] Hall B.C., *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, Springer, 2003.

[3] Rossmann W., *Lie Groups: An Introduction Through Linear Groups*, Oxford University Press, 2002.

[4] Simon B., *Representations of Finite and Compact Groups*, American Mathematical Society, 1996.

[5] Jacobson N., *Lie Algebras*, John Wiley & Sons, 1962.

[6] Humphreys J.E., *Introduction to Lie Algebras and Representation Theory*, Springer, 1972.

[7] Bourbaki N., *Lie Groups and Lie Algebras (Chapters 4-6)*, Springer, 2002.

[8] Baker A., *Matrix Groups: An Introduction to Lie Group Theory*, Springer, 2002.